Meta-equilibrium state of multi-species ambipolar diffusion and its relevance to Polar Mesospheric Summer Echoes

P. M. Bellan

*Applied Physics, MC 128-95 Caltech, Pasadena CA 91125, USA; pbellan@caltech.edu*

Abstract

Normal mode analysis of Hill’s three-species ambipolar diffusion equations [Hill, R.J. (1978), JGR-Space Phys. 83, 989] for an assumed spatially periodic inhomogeneity yields a fast and a slow time scale. For Fourier wavenumbers relevant to Bragg backscattering responsible for Polar Mesospheric Summer Echoes (PMSE) the fast time scale is of the order of milliseconds and the slow time scale is of the order of 10’s of seconds or longer. When initial conditions are taken into account, the appropriate linear combination of fast and slow solutions manifests a relaxation on the fast time scale to a meta-equilibrium state which persists for times of the order of the slow time scale. Once this fast-slow interaction structure leading to the meta-equilibrium state is identified, the meta-equilibrium state can be determined using simple algebraic formulae rather than having to solve the time-dependent differential equations. The relaxed state can be given an intuitive interpretation as the resting point of a pseudo-particle falling into a valley along a track in a two-dimensional pseudo-space. The coordinates of the pseudo-particle are the perturbed densities in Hill’s coupled diffusions equations. The location of the resting point at the bottom of the valley provides the magnitude of the electron perturbation at the Bragg wavelength, i.e., the quantity responsible for PMSE. Application of the algebraic solutions to the PMSE problem reveals the possibility of both overshoots and undershoots in the presence of electron heating by HF waves. Overshoots occur when the ratio of aerosol-bound electrons to gas-phase electrons is small while undershoots occur in the opposite limit.

Key words: PMSE, ambipolar diffusion, noctilucent, electron heating

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1 Introduction and context

Bragg backscattering is presumed (Rapp et al., 2008) to be the mechanism responsible for polar summer mesospheric echoes (PMSE). The power content of a Bragg backscattered signal is proportional to the portion of the turbulent power spectrum of the electron density at the Bragg wavenumber, i.e.,

\[ | \int n_e(z) \exp(ik_Bz)dz \|^2 \]

where \( n_e(z) \) is the vertical electron density profile, \( k_B = 4\pi/\lambda \) is the Bragg wavenumber, and \( \lambda \) is the wavelength of the incident electromagnetic wave (Booker and Gordon, 1950; Landau and Lifshitz, 1960; Boyd and Sanderson, 1960; Hutchison, 1987; Scales, 2004).

In order for Bragg backscattering to cause PMSE, the mesosphere must have vertical electron density inhomogeneities having periodicity on the scale of the Bragg wavelength \( \lambda_B = \lambda/2 \sim 0.5 \) to 3 meters (Rapp et al., 2008). Since \( \lambda_B \) is in the viscous subrange of air density turbulence and since electron diffusivity is much larger than air diffusivity, electrons cannot be a direct tracer for air turbulence because any electron density structure should be completely smoothed out by diffusion (Rapp et al., 2008). Current thinking is that PMSE results from a coupled process whereby long-lived vertically periodic spatial inhomogeneities of much more slowly diffusing charged aerosol particles imprint and maintain a corresponding long-lived vertically periodic spatial inhomogeneity at the Bragg wavelength on the electron density. Because the electron Bragg component is an imprint of the aerosol Bragg component, the effective diffusivity of the electron Bragg component becomes the same as the very low diffusivity of the aerosol Bragg component.

The mesospheric combination of charged aerosol particles, electrons, and ions in a background of neutral molecules constitutes a dusty plasma (Verheest, 2000; Shukla and Mamun, 2002) which is very weakly ionized (ionization fraction \( \sim 10^{-10} \)). Because \( \lambda_B \) greatly exceeds the nominal \( 10^{-2} \) m Debye length of this plasma, the plasma should remain quasi-neutral during the imprinting process. Quasi-neutrality constrains the diffusion to be ambipolar, i.e., quasi-static electric fields must develop that create forces on the various charged species in just such a way for the combination of diffusion and mobility of the various species to maintain neutrality. Hill (1978) first investigated this three-component ambipolar diffusion problem and others have since extended his analysis. Hill demonstrated that having three species substantially changes

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1 Some authors (e.g., Rapp and Lübken (2000) and La Hoz et al. (2006)) have used single Gaussian bumps as the elemental representation of Bragg scattering but representation using a single spatially localized scattering element misses the critical feature that Bragg scattering power is proportional to the square of the number of spatially distributed periodic scattering elements so that, for example, a thousand periodic scattering elements produces a million times as much scattering power as a single scattering element.
the nature of ambipolar diffusion from the classic two-species situation. He showed that the problem reduces to two coupled linearized partial differential equations. Hill examined the temporal decay of an initial quasi-neutral Gaussian density perturbation for the situation where the aerosol particles are negatively charged and found that the electrons would become spatially anti-correlated with the aerosol perturbation, i.e., the aerosol spatial profile would become imprinted on the electron spatial profile. Hill’s work was done prior to the discovery of PMSE (Ecklund and Balsley, 1981) and so was not applied to the PMSE problem.

A basic parameter relevant to this problem is the measure of the extent to which electrons are free or attached to aerosol particles (Cho et al., 1992; Rapp et al., 2002). This measure, known as the Havnes parameter, is

\[ \Lambda = \frac{Z_a \bar{N}_a}{\bar{N}_e} \]  

(1)

where \( Z_a \) is the number of electrons attached to an aerosol particle, \( \bar{N}_a \) is the spatially-averaged aerosol density and \( \bar{N}_e \) is the spatially-averaged gas-phase electron density. \( \Lambda = 0 \) thus corresponds to no electrons being attached to the aerosol particles, \( \Lambda = 1 \) corresponds to half the electrons being free and half attached, and \( \Lambda = \infty \) corresponds to the limit of all electrons attached and none free (i.e., bite-out).

Cho et al. (1992) proposed that Hill’s ambipolar diffusion equations could explain how aerosol spatial periodicity is imprinted on the electron density and cause the electrons to produce Bragg back scattering of radar waves, i.e., PMSE. By solving the ambipolar diffusion equations numerically for an assumed periodic aerosol spatial inhomogeneity, Cho et al. defined and calculated an effective electron diffusivity. Cho et al. found that this effective electron diffusivity abruptly decreased by orders of magnitude when \( \Lambda > 1.2 \) (see Fig.1(a) and (b) of Cho et al.) and argued that this abrupt decrease could explain PMSE. Cho et al. assumed that all species remained at the same nominal 150 K temperature.

In an attempt to explain the observations by Chilson et al. (2000) that application of a 30 s duration high-power 4-5 MHz wave (HF wave) to the mesosphere sharply reduced the amplitude of 224 MHz PMSE, Rapp and Lübken (2000) extended the analysis by Cho et al. (1992) to take into account electron heating by the HF wave. This extended analysis by Rapp and Lübken (2000) involved modifying the Hill (1978) equations to incorporate a presumed twenty-fold increase in electron temperature as a result of HF heating. They solved the equations numerically for the same Gaussian bump initial condition used by Hill, assumed \( \Lambda = 2 \) in order to satisfy Cho’s \( \Lambda > 1.2 \) criterion and found that the spatial modulation of electron density by the aerosol spatial inhomogeneity was substantially reduced when the electrons were heated. They noted
that heating the electrons would convert gas-phase electrons to aerosol-bound electrons as a consequence of heating increasing the flux of electrons impacting aerosol grains. However, the effect of aerosol charging was only discussed qualitatively and it was noted that increasing the aerosol charge should both increase the ability of aerosols to modulate electrons and decrease the number of electrons available to be modulated. Later, Rapp and Lübken (2003) concluded that Cho’s criterion that PMSE occurs only when $\Lambda > 1.2$ was spurious because the criterion resulted from use of an unphysical initial condition where the electrons were assumed to be correlated with the charged aerosol particles rather than anti-correlated.

Like Rapp and Lübken (2000), La Hoz et al. (2006) invoked the evolution of a Gaussian bump as representative of the structure responsible for PMSE. La Hoz et al. (2006) prescribed a Gaussian bump electron inhomogeneity as an initial condition and then used charge neutrality to determine associated initial dust and ion density inhomogeneities. However, La Hoz et al. did not specify whether their assumed initial Gaussian electron bump was correlated or anti-correlated with the aerosol density. This lack of specification constitutes an important ambiguity in the analysis because both correlated and anti-correlated assumptions are viable initial conditions; this is because quasi-neutrality can be achieved for any assumed aerosol and electron density distributions by suitable adjustment of the positive ion density distribution. Like Hill, La Hoz et al. noted that the normal mode problem incorporates two effective time scales each with a dependence $\sim (k_B^2 D)^{-1}$ where $D$ is a diffusion parameter that is the large or small root of a determinant associated with the normal mode analysis of Hill’s coupled diffusion equations. La Hoz et al. argued that the correct width for the Gaussian bump should be $k_B^{-1}$ and not $\lambda_B$ as used by Rapp and Lübken (2000). This produces a $(2\pi)^2 \simeq 40$ discrepancy between the effective diffusion calculated by La Hoz et al. (2006) and that calculated by Rapp and Lübken (2000).

Havnes et al. (2003, 2004) modeled the effect of heating using a very different perspective from that of Rapp and Lübken (2000) and La Hoz et al. (2006). Havnes et al. (2003, 2004) predicted and then interpreted observations of the ‘overshoot effect’ whereby PMSE strength after heating is turned off can be briefly larger than the PMSE strength before heating was turned on. Rather than use Hill’s ambipolar diffusion equations, Havnes et al. (2004) invoked a Boltzmann equilibrium model. Also, rather than use a Gaussian bump, Havnes et al. (2004) considered how the assumed Boltzmann equilibrium depended on the electron temperature.

Scales (2004) used a Particle-in-Cell (PIC) code with immobile aerosol particles, mobile positive ions, and mobile electrons to model how electron heating affects the Bragg-relevant component of the electron density inhomogeneity. The code included charging currents, electron-ion production by photo-
ionization of neutral gas molecules, and electron-ion recombination. Scales (2004) plotted PIC results for the spatially dependent equilibrium densities of all species (i) prior to heating, (ii) when heating was on, and (iii) after heating was turned off. These plots showed that the electron spatial inhomogeneity decreased by more than a factor of ten during electron heating, but no significant overshoot or undershoot was observed.

Recently the author proposed that PMSE at a bite-out might be produced by radar backscattering directly from charged aerosol grains rather than from free electrons on the grounds that at a bite-out there are almost no free electrons to do the scattering (Bellan, 2008). This proposal has been controversial (Rapp and Lübken, 2009; Bellan, 2010; Rapp and Lübken, 2010) and the mechanism for direct radar backscattering from charged aerosol grains remains uncertain (Bellan, 2010). The present paper does not address this controversy and instead considers only the conventional situation where there is no bite-out and PMSE results from backscattering from free electrons.

The purpose of this paper is to show that for PMSE-relevant densities \( \sim \cos(k_Bz) \), the solutions to Hill’s coupled diffusion equations relax on the fast time scale to a long-lived meta-equilibrium state and that consideration of this state enables derivation of a simple algebraic formula giving the PMSE-relevant Bragg electron density spatial distribution. The predictions of this simple algebraic formula are consistent with the PIC numerical results reported by Scales (2004). These formula could not have been determined using Gaussian bump arguments nor using any other solitary structure. In fact, Gaussian bump evolution will be shown to be irrelevant to the Bragg scattering problem because it will be shown that the only initial condition that matters for producing PMSE is the Bragg Fourier component of the aerosol density. Once this is given, the electron density relevant to PMSE is determined regardless of the electron density initial condition; the system ‘remembers’ the aerosol initial condition but ‘forgets’ the electron initial condition. Rapp and Lübken (2000), Cho et al. (1992), and La Hoz et al. (2006) each specified electron density initial conditions without explicitly specifying the Bragg component aerosol density initial conditions, and then followed the electron density evolution. For a given Bragg-relevant aerosol spatial distribution, the configuration always relaxes to have the same meta-equilibrium perturbed electron density state no matter what choice was made for the initial perturbed electron distribution.
2 Meta-equilibrium solution to Hill’s equations

2.1 Arranging equations to allow asymptotic analysis

Following the procedure described in Hill (1978), the electron, positively charged ion, and aerosol densities are linearized as

\[ N_e = \bar{N}_e + n_e \]  
\[ N_+ = \bar{N}_+ + n_+ \]  
\[ N_a = \bar{N}_a + n_a \]

and equations are derived to characterize the time evolution of

\[ n_{\oplus} = n_+ + n_a \]  
\[ n_e = n_+ - n_a . \]

Once \( n_{\oplus} \) and \( n_e \) have been determined \( n_a \) and \( n_+ \) can be solved for using the inversion relations

\[ n_a = \frac{n_{\oplus} - n_e}{2} \]  
\[ n_+ = \frac{n_{\oplus} + n_e}{2}. \]

Equation 6 is the charge neutrality condition and for now \( Z_a = 1 \) is assumed, a constraint to be relaxed later. Consideration of ambipolar diffusion results in the two coupled equations [Eq.(1) in Rapp and Lübken (2000)]

\[ \frac{\partial n_{\oplus}}{\partial t} = \frac{D_+ + D_-}{2} \nabla^2 n_{\oplus} + \left( \frac{D_+ T_e}{T_n} + \frac{D_+ - D_-}{2} \left( 1 + 2\Lambda T_e \right) \right) \nabla^2 n_e \]
\[ \frac{\partial n_e}{\partial t} = \frac{D_+ - D_-}{2} \nabla^2 n_{\oplus} + \left( \frac{D_+ T_e}{T_n} + \frac{D_+ + D_-}{2} \left( 1 + 2\Lambda T_e \right) \right) \nabla^2 n_e \]

where \( D_+ \) is the positive ion diffusion coefficient and \( D_- \) is the negatively charged aerosol diffusion coefficient.

For the situation of interest here where the aerosol particles can be considered as hard spheres much larger than a single molecule, Cho et al. (1992) give in their Eq.6 the aerosol diffusion coefficient to be

\[ D_- = \frac{3}{8} \sqrt{\frac{\kappa T}{2\pi m_a N_n r_a^2}} . \]

For nominal mesospheric parameters ( neutrals dominantly nitrogen molecules, \( T = 150 \) K, \( N_n = 2 \times 10^{20} \) m\(^{-3}\) ) Eq.11 gives \( D_- = 1.6 \times 10^{-3} \) m\(^2\) s\(^{-1}\) for an
aerosol radius \( r_a = 10 \text{ nm} \) and \( D_- = 6.3 \times 10^{-5} \text{ m}^2 \text{ s}^{-1} \) for an aerosol radius \( r_a = 50 \text{ nm} \).

The positive ion (i.e. nitrogen ion) diffusion coefficient is

\[ D_+ = \frac{\kappa T}{m_+ \nu_+} = \frac{\kappa T}{m_+ N_n \sigma u_+} \]  

where \( \sigma \) is the cross section for nitrogen ions colliding with nitrogen molecules and \( u_+ = \sqrt{2\kappa T/m_+} \) is the ion thermal velocity. By extrapolating Fig. 1 of Phelps (1990), Robertson (2007) noted that \( \sigma \) scales as the square root of temperature in the relevant temperature range so \( D_+ \) is approximately independent of temperature in this range. Using \( \sigma u_+ = 1.3 \times 10^{-15} \text{ m}^3 \text{ s}^{-1} \) at \( \kappa T = 0.1 \text{ eV} \) from Robertson (2007) and \( N_n = 2 \times 10^{20} \text{ m}^{-3} \) gives \( D_+ = 2.6 \text{ m}^2 \text{ s}^{-1} \). Thus, \( D_-/D_+ \sim 10^{-3} \) if \( r_a = 10 \text{ nm} \) and \( D_-/D_+ \sim 10^{-5} \) if \( r_a = 50 \text{ nm} \). The essential point here is that aerosol diffusivity is orders of magnitude smaller than positive ion diffusivity.

Because Eqs. 9 and 10 are linear equations, they apply independently to each spatial Fourier component. Since only the spatial Fourier component \( \sim \cos(k_B x) \) contributes to Bragg scattering, it is appropriate to restrict consideration to perturbations \( \sim \cos(k_B x) \) in which case Eqs. 9 and 10 reduce using \( \nabla^2 \rightarrow -k_B^2 \) to ordinary differential equations. By defining

\[ \tau = \frac{k_B^2 D_+ t}{2} \]
\[ \varepsilon = \frac{D_-}{D_+}, \]

Eqs.9 and 10 become

\[ -\frac{\partial n_\oplus}{\partial \tau} = (1 + \varepsilon) n_\oplus + \left( \frac{2T_e}{T_n} + (1 - \varepsilon) \left( 1 + 2\Lambda \frac{T_e}{T_n} \right) \right) n_e \]  

\[ -\frac{\partial n_e}{\partial \tau} = (1 - \varepsilon) n_\oplus + \left( \frac{2T_e}{T_n} + (1 + \varepsilon) \left( 1 + 2\Lambda \frac{T_e}{T_n} \right) \right) n_e \]

where \( n_\oplus \) and \( n_e \) now refer to only the inhomogeneities having spatial dependence \( \cos(k_B x) \) and \( \varepsilon \) is a small parameter of the order of \( 10^{-5} - 10^{-3} \) determined by the aerosol particle size. The smallness of \( \varepsilon \) allows for an asymptotic analysis which reveals the essential structure of the solutions to Eqs. 15 and 16; this structure is determined in the next section.

2.2 Normal mode analysis and identification of meta-equilibrium

We assume that either \( T_e/T_n = 1 \) or \( T_e/T_n \gg 1 \), i.e., electron heating is either off or on. The diffusive nature of Eqs. 15 and 16 suggests that \( n_\oplus \) and
$n_e$ should have an exponentially decaying time dependence (Hill, 1978). We therefore assume that $n_{\oplus}$ and $n_e$ have an $\exp(-\gamma \tau)$ time dependence when $T_e/T_n$ is constant in time so that Eqs. 15 and 16 reduce to two homogeneous algebraic equations in $n_\oplus$ and $n_e$. Because the equations are homogeneous, their determinant must vanish, i.e.,

$$
(1 + \varepsilon - \gamma) \left( 2 \frac{T_e}{T_n} + (1 + \varepsilon) \left( 1 + 2 \Lambda \frac{T_e}{T_n} \right) - \gamma \right) - \left( 2 \frac{T_e}{T_n} + (1 - \varepsilon) \left( 1 + 2 \Lambda \frac{T_e}{T_n} \right) \right) (1 - \varepsilon) = 0.
$$

(17)

Equation 17 can be arranged as a quadratic in $\gamma$

$$
\gamma^2 - 2\gamma \left( 1 + \frac{T_e}{T_n} (1 + \Lambda) \right) + 4\varepsilon \left( 1 + \frac{T_e}{T_n} (1 + 2\Lambda) \right) = 0
$$

(18)

where only the leading order terms in $\varepsilon$ have been retained. The procedure of calculating the determinant given by Eq. 17 has been discussed previously by Hill and by La Hoz et al.; what is new here is that the smallness of $\varepsilon$ is exploited to obtain asymptotic analytic solutions that can be physically interpreted.

Because $\varepsilon$ is small, the two roots of Eq. 18 are well-separated, i.e., the large root is much larger than the small root. The large (fast) root denoted $\gamma_f$ is obtained by balancing the first two terms in Eq. 18, i.e.,

$$
\gamma_f = 2 \left( 1 + \frac{T_e}{T_n} (1 + \Lambda) \right).
$$

(19)

The small (slow) root denoted $\gamma_s$ is obtained by balancing the last two terms in Eq. 18, i.e.,

$$
\gamma_s = 2\varepsilon \frac{1 + \frac{T_e}{T_n} (1 + 2\Lambda)}{1 + \frac{T_e}{T_n} (1 + \Lambda)}.
$$

(20)

The slow root is $O(\varepsilon)$ smaller than the fast root; this scale difference will now be used to show that the relevant state of interest is a meta-equilibrium that can be determined analytically. Furthermore, it will be shown that this meta-equilibrium state has a one-to-one dependence on the $n_\oplus$ initial condition, but is independent of the $n_e$ initial condition.

Both $n_\oplus$ and $n_e$ must be linear combinations of $e^{-\gamma_f \tau}$ and $e^{-\gamma_s \tau}$ but $n_\oplus$ and $n_e$ have different initial conditions. Thus if $\psi$ represents either $n_\oplus$ or $n_e$, $\psi$ must be of form

$$
\psi = Ae^{-\gamma_f \tau} + Be^{-\gamma_s \tau}
$$

(21)

where $A$ and $B$ are coefficients to be determined and where $\psi = \psi_0$ at $\tau = 0$. Equations 15 and 16 show that, to lowest order in $\varepsilon$, the derivative of $\psi$ satisfies
the initial condition

\[
- \frac{\partial \psi}{\partial \tau} \bigg|_{\tau=0} = n_{\oplus}(0) + \left(1 + 2 \frac{T_e}{T_n} (1 + \Lambda)\right) n_e(0). \tag{22}
\]

Thus, \(A\) and \(B\) are determined by the two initial condition equations

\[
A + B = \psi_0 \tag{23}
\]

\[
\gamma_f A + \gamma_s B = F \tag{24}
\]

where

\[
F = n_{\oplus}(0) + \left(1 + 2 \frac{T_e}{T_n} (1 + \Lambda)\right) n_e(0). \tag{25}
\]

Solution of Eqs. 23 and 24 for \(A\) and \(B\) give

\[
\psi(\tau) = \frac{(F - \gamma_s \psi_0) e^{-\gamma_f \tau} + (\gamma_f \psi_0 - F) e^{-\gamma_s \tau}}{\gamma_f - \gamma_s}. \tag{26}
\]

Since \(\gamma_f\) is larger than \(\gamma_s\) by \(O(\varepsilon^{-1})\), the term containing \(\exp(-\gamma_f \tau)\) will quickly decay; furthermore for times \(\tau\) substantially less than \(\gamma_s^{-1}\) it is seen that \(\exp(-\gamma_s \tau) \simeq 1\). Thus for times \(\gamma_f^{-1} \ll \tau \ll \gamma_s^{-1}\) the solution to Eqs. 15 and 16 is approximately constant in time and given by

\[
\psi(\tau) \simeq \frac{\gamma_f \psi_0 - F}{\gamma_f}; \tag{27}
\]

this is the meta-equilibrium state. Since it is the electron Bragg component that is responsible for PMSE, we now focus attention on the electron Bragg Fourier component so \(\psi\) becomes \(n_e\). Substituting for \(\gamma_f\) and \(F\) using Eqs. 19 and 25, and then using Eq. 7, Eq. 27 reduces to

\[
n_e(\tau) \simeq -\frac{n_a(0)}{1 + \frac{T_e}{T_n} (1 + \Lambda)} \quad \text{for} \quad \gamma_f^{-1} \ll \tau \ll \gamma_s^{-1}. \tag{28}
\]

Equation 28 shows that the PMSE-relevant meta-equilibrium electron density depends on the aerosol particle initial condition and not on the initial condition of the electron spatial inhomogeneity.

The system is approximately time-independent during the meta-equilibrium interval \(\gamma_f^{-1} \ll \tau \ll \gamma_s^{-1}\) corresponding to physical times

\[
\frac{1}{k_B^2 D_+} \ll t \ll \frac{1}{k_B^2 D_-}. \tag{29}
\]

The meta-equilibrium interval owes its existence to the large \(D_+/D_-\) ratio. The slow decay quantity \(\gamma_s \tau\) can be expressed in terms of un-normalized time
and so is approximately independent of both $T_e/T_n$ and $\Lambda$. The time at which $\gamma_s \tau = 1$ is

$$t_s = \frac{\lambda_B^2}{4\pi^2 D_-};$$  \hspace{1cm} (31)$$

this is the time for aerosol grains to diffuse a distance corresponding to a Bragg wavelength $\lambda_B$. It is seen that $t_s = 7$ s for aerosols with $r_a = 10$ nm and $t_s = 180$ s for aerosols with $r_a = 50$ nm (assuming Bragg wavelength $\lambda_B = 0.67$ m). We note that $\gamma_f \tau \gg 1$ if $\gamma_s \tau = 1$. If the aerosol radius is $r_a = 50$ nm, the Bragg perturbation would persist for the nominal 30 second interval relevant to HF heating experiments discussed by Rapp and Lübken (2009).

However, if the aerosol radius were 10 nm, Bragg-scale spatial inhomogeneities of aerosols, electrons, and positive ions would all be completely attenuated in the nominal 30 second time interval. One might argue that if there were a continuous source driving the turbulence, then Bragg scattering from 10 nm aerosols could exist for durations much longer than the 7 s diffusion time of 10 nm aerosols. However, Lübken et al. (2002) reported PMSE occurrence in situations where there is no measurable air turbulence so it is not tenable to argue that 10 nm aerosols previously modulated by air turbulence could provide PMSE for the nominal 30 second time interval or for longer intervals. Another possibility, of course, is that the aerosol turbulence is not driven by air turbulence at all, but is instead caused by something else, for example a current-driven turbulent dusty plasma wave as proposed by D’Angelo (2005).

Let us thus assume that $r_a \geq 50$ nm and that the heating experiment is for times sufficiently short that $\gamma_s \tau \ll 1$. This means that the aerosol particles have not undergone significant diffusion and therefore could in principle imprint their spatial density distribution on that of the electrons. The fast decay quantity $\gamma_f \tau$ is unity when

$$t_f = \frac{1}{k_B^2 D_+ \left(1 + \frac{T_e}{T_n} (1 + \Lambda) \right)}.$$

(32)$$

Since $t_f \ll 5$ ms for all values of $T_e/T_n$ and all values of $\Lambda$, it is clear that $t_f$ is much shorter than the resolution time of the HF heating experiments. The factor $e^{-\gamma_f \tau}$ can therefore be considered to be essentially a switch that is either on or off (value unity or zero). The situation $e^{-\gamma_f \tau} = 1$ corresponds to an initial condition situation where $t = 0$ and the situation $e^{-\gamma_f \tau} = 0$ corresponds to any time more than 5 ms after $t = 0$.

Any initial distribution with the same value of $n_a(0)$ will always decay to the same final meta-equilibrium state for times satisfying Eq.29. For example, the initial conditions $n_e = n_0$, $n_\oplus = (1 + 2\Lambda) n_0$ and the completely differ-
ent initial conditions $n_e = -5n_0$, $n_\oplus = (-4 + 2\Lambda)n_0$ will both decay to the
same metastable state since both initially have $n_e(0) - n_\oplus(0) = -2\Lambda n_0$ which
implies that both sets of initial conditions have $n_a = \Lambda n_0$. Prescribing initial
conditions for $n_e$ and $n_\oplus$ as in the previous sentence has the effect of causing
the initial condition for $n_a$ to depend on $\Lambda$. However, because the density
of aerosol particles does not in reality depend on the extent to which they
are charged, imposing this dependence of $n_a(0)$ on $\Lambda$ is spurious and leads to
misleading predictions for the scaling of $n_e$.

Appendix A shows that when the fast decay is followed in detail, the system
always relaxes to the meta-equilibrium state given by Eq.28.

3 Meta-equilibrium interpreted as relaxation of a fictitious particle
to a minimum energy state

Armed with knowledge of the existence of the metastable equilibrium we can
now skip the details of the normal mode analysis. Instead we simply consider
times where $\gamma_s\tau \ll 1$ so that the aerosol particles have not moved. This
 corresponds to assuming infinitely massive aerosol grains, i.e., to assuming
$\varepsilon = 0$. Equations 15 and 16 then reduce to the simple system

$$\frac{-\partial n_\oplus}{\partial \tau} = n_\oplus + an_e$$  \hspace{1cm} (33)
$$\frac{-\partial n_e}{\partial \tau} = n_\oplus + an_e$$  \hspace{1cm} (34)

where

$$a = 1 + 2(\Lambda + 1) \frac{T_e}{T_n}. \hspace{1cm} (35)$$

By defining the vector $\mathbf{R} = \{X, Y\}$ where $X = n_\oplus/\sqrt{a}$ and $Y = n_e$ Eqs.33
and 34 can be expressed as

$$\frac{d\mathbf{R}}{d\tau} = -\nabla U(X, Y)$$  \hspace{1cm} (36)

where $U(X, Y)$ is an effective potential energy given by

$$U(X, Y) = \frac{X^2}{2} + \frac{aY^2}{2} + \sqrt{a}XY. \hspace{1cm} (37)$$

The steady-state solution of Eqs.33 and 34 corresponds to a fictitious particle
with coordinates $\{X, Y\}$ relaxing to a position where the effective potential
energy $U(X, Y)$ is at a minimum. The particle can be imagined as sliding
down the slope of the $U(X, Y)$ ‘hill’ with velocity proportional to the local
gradient of the hill and then coming to rest at the bottom of the hill. Once
the particle arrives at this local minimum of $U(X,Y)$, it stops; this is the
meta-equilibrium state.

Subtracting Eq.33 from Eq.34 and integrating gives the constraint

$$ n_{\oplus} = n_e + c $$

(38)

where $c$ is a constant determined by initial conditions; this is true even if
$a$ depends on time, i.e., even if $T_e/T_n$ is non-constant. This integration step
thus both takes the fast time dependence into account and describes situa-
tions where the electron temperature might be varying as in an HF heating
experiment. Using Eq.7 it is seen that

$$ c = 2n_a $$

(39)

where the aerosol density $n_a$ is unchanging because $\gamma_s \tau \ll 1$. Requiring the
system to satisfy Eq.38 can be visualized as prescribing a track $Y = \sqrt{a}X - c$
in the $\{X,Y\}$ plane on which the particle must remain as it slides down the
$U(X,Y)$ hill to its resting spot where $U$ is a minimum. Starting the particle
at any point on a given track will always result in the particle coming to rest
at the same final position. Thus, the specific initial condition matters only so
far as the track is determined; on each track there are an infinite number of
initial conditions and each of these will lead to the same relaxed state. Each
track corresponds to a different value of $n_a$ and each position on a given track
corresponds to choice of a different initial condition for $n_e$.

The system thus relaxes to the meta-equilibrium state (i.e., local minimum of
$U(X,Y)$) given by

$$ n_{\oplus} + an_e = 0 $$

(40)

subject to Eq.38 which constrains the particle to remain on the track deter-
mined by the value of $n_a$. Substitution for $n_{\oplus}$ in Eq.40 using Eq.38 gives

$$ n_e = -\frac{2n_a}{1 + a} $$

(41)

or using Eq.35

$$ n_e = -\frac{n_a}{1 + (1 + \Lambda) \frac{T_e}{T_n}} $$

(42)

which is valid for any value of $T_e/T_n$. The minus sign shows that the electrons
always relax to a meta-equilibrium which is is anti-correlated with the aerosol
particles.

For fixed $\Lambda$, the ratio of electron density after heating (i.e., $T_e^h/T_n \gg 1$)
compared to the density before heating (i.e., $T_e/T_n = 1$) is
\[
\frac{n_e^h}{n_e^c} = \frac{\Lambda + 2}{1 + \frac{T_e}{T_n}(\Lambda + 1)}.
\] (43)
For Rapp and Lübben (2000)’s assumption of $\Lambda = 2$ and $T_e/T_n = 20$, Eq.43 predicts the ratio $n_e^h/n_e^c = 328^{n_e^h/n_e^c} = \Lambda + 2 \left(1 + \frac{T_e}{T_n}(\Lambda + 1)\right)$.

For Rapp and Lübben (2000)’s assumption of $\Lambda = 2$ and $T_e/T_n = 20$, Eq.43 predicts the ratio $n_e^h/n_e^c = 6.6 \times 10^{-2}$. Since Bragg backscattering scales as $n_e^2$ (Booker and Gordon, 1950; Landau and Lifshitz, 1960; Boyd and Sanders, 1960; Hutchison, 1987), PMSE power should be reduced by the factor $(n_e^h/n_e^c)^{-2} \approx 230$ when the electrons are hot compared to when they are cold. Since the actual observed reduction in PMSE is only about a factor of two, this indicates that Rapp and Lübben (2000)’s model is not in excellent agreement with the observations. Furthermore, accumulation of electrons on aerosols would tend to increase $\Lambda$ which would make the disagreement worse. However, because increase of $\Lambda$ is also associated with an increase in $Z_a$, the non-constant $\Lambda$ situation has to be investigated with greater care. This is done in the next section.

4 Meta-equilibrium for time-dependent $\Lambda$

The previous analysis showed that when $T_e/T_n$ is changed, the system adjusts on the rapid $\gamma_f$ time-scale to a new meta-equilibrium. We now take into account time-dependence of $\Lambda$. To do this the associated increase in $Z_a$ and decrease in $\bar{N}_e$ must also be taken into account; i.e., the Rapp and Lübben (2000) assumption that $Z_a$ was fixed at unity is now inadequate. The Hill equations must therefore be generalized to allow for $T_e/T_n$, $\Lambda$, and $Z_a$ all to change. However, the realization that the system always relaxes into a meta-equilibrium enables analysis of this much more complex system without great difficulty.

In order to generalize Hill’s equations, we start with force-balance for each species
\[
0 = N_\alpha q_\alpha \mathbf{E} - \kappa T_\alpha \nabla N_\alpha - N_\alpha m_\alpha \nu_\alpha \mathbf{v}_\alpha
\] (44)
where $\nu_\alpha$ is the momentum loss collision frequency due to collisions with all species different from $\alpha$ (e.g., $\nu_e$ would represent the rate at which electrons are scattered by 90° and so lose their directed momentum due to collisions with ions, aerosol particles, and neutrals).

By defining mobilities $\mu_\alpha = q_\alpha/m_\alpha \nu_\alpha$, diffusion coefficients $D_\alpha = \kappa T_\alpha/m_\alpha \nu_\alpha$, and particle fluxes $\Gamma_\alpha = N_\alpha \mathbf{v}_\alpha$, Eq.44 can be written as
\[
N_\alpha \mu_\alpha \mathbf{E} = D_\alpha \nabla N_\alpha + \Gamma_\alpha.
\] (45)
Multiplying Eq. 45 by \( q_\alpha \), invoking flux quasineutrality to give \( \sum_\alpha q_\alpha \Gamma_\alpha = 0 \) and then solving the resultant equation gives the ambipolar electric field

\[
E = \frac{\sum_\alpha q_\alpha D_\alpha \nabla N_\alpha}{\sum_\alpha N_\alpha q_\alpha \mu_\alpha}.
\]

(46)

Because the electron diffusion and mobility coefficients are so much larger than those of the other species, the electron terms in the numerator and denominator of Eq. 46 dominate the other terms in which case Eq. 46 reduces to

\[
E \simeq \frac{D_e \nabla N_e}{\mu_e N_e} = \frac{\kappa T_e \nabla N_e}{q_e N_e}.
\]

(47)

It should be noted, however, that at sufficiently strong bite-out (i.e., \( N_e \to 0 \)), the assumption that electron diffusion and mobility dominates diffusion and mobility of other species in Eq. 46 will become inappropriate for lack of electrons.

Equation 45 can be expressed using Eq. 47 as

\[
\Gamma_\alpha = -D_\alpha \nabla N_\alpha + D_\alpha \frac{N_\alpha q_\alpha T_e}{N_e q_e T_a} \nabla N_e.
\]

(48)

We define

\[
N_\oplus = N_+ + N_a
\]

(49)

where \( N_+ \) is the density of positive ions and \( N_a \) is the aerosol density while quasineutrality gives

\[
N_e = N_+ - Z_a N_a.
\]

(50)

The continuity equations for positive ions and for aerosol particles are respectively

\[
\frac{\partial N_+}{\partial t} = -\nabla \cdot \Gamma_+
\]

(51)

\[
\frac{\partial N_a}{\partial t} = -\nabla \cdot \Gamma_a.
\]

(52)

The time derivative of Eq. 49 combined with Eqs. 51 and 52 gives

\[
\frac{\partial N_\oplus}{\partial t} = -\nabla \cdot \Gamma_+ - \nabla \cdot \Gamma_a
\]

(53)

while the time derivative of Eq. 50 combined with Eqs. 51 and 52 gives

\[
\frac{\partial N_e}{\partial t} = -\nabla \cdot \Gamma_+ - N_a \frac{\partial Z_a}{\partial t} + Z_a \nabla \cdot \Gamma_a
\]

(54)

showing an explicit depletion of free electrons as the aerosol charge \( Z_a \) increases.
It was shown in Sec. 3 that because the aerosol flux $\Gamma_a$ is of order $\varepsilon$ it can be dropped for the time scale of interest; i.e., since $\varepsilon$ is small, the aerosol particles can be considered to be stationary on the time scale of interest in which case $\Gamma_a = 0$ (this elimination of aerosol motion from consideration during the meta-equilibrium state contrasts with La Hoz et al. where all terms were kept, the coupled diffusion equations were not restricted to a Bragg mode, and the existence of the meta-equilibrium state was not identified; however as will be seen, elimination of aerosol motion from consideration is consistent with the analysis by Scales (2004)). Equations 53 and 54 thus become

\[
\frac{\partial N_\oplus}{\partial t} = -\nabla \cdot \Gamma_+ \tag{55}
\]

\[
\frac{\partial}{\partial t} (N_e + N_a Z_a) = -\nabla \cdot \Gamma_+ . \tag{56}
\]

Subtraction of these equations gives

\[
\frac{\partial}{\partial t} (N_\oplus - (N_e + N_a Z_a)) = 0; \tag{57}
\]

which effectively identifies a parameter that must be conserved during time-dependent processes. Since we are interested in the Bragg-component of the electron density, we linearize Eq. 57

\[
\frac{\partial}{\partial t} (n_\oplus - (n_e + (N_a Z_a)_1)) = 0 \tag{58}
\]

where $(N_a Z_a)_1 = n_a Z_a + N_a z_a$ and $z_a$ is the amplitude of the Bragg spatial component of the aerosol charge. Integration of Eq. 58 gives the Bragg-relevant conserved quantity

\[
(n_\oplus - (n_e + (N_a Z_a)_1)) = c \tag{59}
\]

where $c$ is a constant to be determined.

Linearization of Eqs. 49 and 50 give

\[
n_\oplus = n_+ + n_a \tag{60}
\]

\[
n_e = n_+ - (Z_a N_a)_1 \tag{61}
\]

and substitution for $n_\oplus$ and $n_e + (N_a Z_a)_1$ in Eq. 59 gives

\[
c = n_a . \tag{62}
\]

Furthermore, the steady-state meta-equilibrium has $\partial/\partial t = 0$, i.e., the right hand sides of Eqs. 55 and 56 must vanish in steady state. Since we are interested in Bragg components, this means that the Bragg component of $\nabla \cdot \Gamma_+$ must
vanish where using $T_+ = T_n$ and $q_+ / q_e = -1$, 

$$\mathbf{\Gamma}_+ = -D_+ \left( \nabla N_+ + \frac{N_+}{N_e} \frac{T_e}{T_n} \nabla N_e \right). \quad (63)$$

Linearizing Eq.63, taking the Bragg component, and setting this component to zero gives

$$n_+ + \frac{\bar{N}_+ T_e}{N_e T_n} n_e = 0. \quad (64)$$

Dividing the spatially averaged form of Eq.50 by $\bar{N}_e$, gives

$$\frac{\bar{N}_+}{\bar{N}_e} = \frac{\bar{N}_e + Z_a N_a}{\bar{N}_e} = 1 + \Lambda \quad (65)$$

where $\Lambda$ is defined by Eq.1 so Eq.64 becomes

$$n_+ + (1 + \Lambda) \frac{T_e}{T_n} n_e = 0. \quad (66)$$

Combining Eqs.59, 60, 62, and 66 gives

$$n_e = -\frac{n_a Z_a + N_a z_a}{1 + (1 + \Lambda) \frac{T_e}{T_n}} \quad (67)$$

the generalization of Eq.42 to the situation where $\Lambda$ and $Z_a$ are not constant.

Because the aerosol potential is much more sensitive to changes in electron temperature than to changes in electron density, we argue that $|N_a z_a| \ll |n_a Z_a|$, i.e., spatial modulation of the aerosol density at the Bragg wavelength is much more important than spatial modulation of the aerosol charge at the Bragg wavelength. Spatial modulation of the aerosol charge could only result from the integrated effect of charging due to a spatially modulated electron density, but this is a weak effect because the aerosol charge is related to the aerosol floating potential which in turn depends mainly on the electron temperature. The electron temperature has no spatial inhomogeneity at the Bragg wavelength. Hence, we conclude that the Bragg-component electron density is

$$n_e \simeq -\frac{n_a Z_a}{1 + (1 + \Lambda) \frac{T_e}{T_n}} \quad (68)$$

which is the generalization of Eq.42 to situations where $Z_a$ and $\Lambda$ are time-dependent. However, as will be seen in Sec.4.1.3, it is possible for $z_a$ to be important in a special situation where the ion flux to the aerosol particles is much larger than the electron flux and this ion flux has a large spatial modulation at the Bragg wavelength; in this situation Eq.67 should be used with the appropriate $z_a$.

Combining Eqs.64 and 68 gives the associated positive ion flux perturbation.
to be
\[ n_+ = n_a Z_a T_e \frac{(1 + \Lambda)}{T_n 1 + (1 + \Lambda) \frac{T_e}{T_n}}. \] (69)

Equation 68 shows that \( n_+ \) depends only on the present values of \( n_a, Z_a, \Lambda, \) and \( T_e/T_n \) and not on any given initial condition for \( n_+ \). It also shows that if no electrons are on the aerosol particles (i.e., \( Z_a = 0 \)) then \( n_+ = 0 \) and if all the electrons reside on the aerosol particles (i.e., \( \Lambda = \infty \)) then \( n_+ = 0 \) also. The former case corresponds to the aerosol particles being unable to spatially modulate the electrons while the latter case corresponds to there being no electrons to be modulated. In particular, the latter case shows, as argued in Bellan (2008, 2010), there should be no PMSE when \( \Lambda \to \infty \), i.e., at bite-out.

\( \Lambda \) has a dependence on \( Z_a \) which is determined as follows: Since increasing the aerosol charge via electron heating does not affect the total amount of negative charge, Eq.50 can be expressed as
\[ \bar{N}_e + Z_a \bar{N}_a = \bar{N}_{e0} + Z_{a0} \bar{N}_a \] (70)
where subscript 0 denotes the initial value before electron heating and \( \bar{N}_a \) is unchanging. By writing \( \bar{N}_{e0} = Z_{a0} \bar{N}_a / \Lambda_0 \) and \( \bar{N}_e = Z_a \bar{N}_a / \Lambda \), Eq.70 can be solved for \( Z_a \) as a function of \( \Lambda \)
\[ Z_a = Z_{a0} \frac{(1 + \frac{1}{\Lambda_0})}{(1 + \frac{1}{\Lambda})}. \] (71)

or, inversely, as \( \Lambda \) as a function of \( Z_a \)
\[ \Lambda = \frac{1}{(1 + \frac{1}{\Lambda_0}) \frac{Z_{a0}}{Z_a} - 1}. \] (72)

These expressions show that for a given initial \( Z_{a0} \), the maximum possible \( Z_a \) that can be attained is \( Z_a = Z_{a0} \left(1 + \frac{1}{\Lambda_0}\right) \) so for example, if \( \Lambda_0 = 2 \), the maximum possible increase in dust charge is 50%, i.e., \( Z_a/Z_{a0} = 1.5 \) when \( \Lambda = \infty \).

Equation 68 together with either Eq.71 or 72 give the meta-equilibrium Bragg-relevant dust density perturbation and can be used to calculate the effect of electron heating on PMSE. A detailed example is given in the next section which shows that these equations give numerical values in reasonable agreement with the PIC code results reported by Scales (2004).
4.1 Comparison of predictions of meta-equilibrium analytic model with Scales (2004) particle-in-cell numerical code

By inserting appropriate values of $Z_a$, $\Lambda$, and $T_e/T_n$ in Eqs.68 and 69 we now show that these equations predict values of $n_e$ and $n_+$ that are consistent with the particle-in-cell calculations reported by Scales (2004). For purposes of making the comparison it should be noted that Scales (2004) uses ‘dust to mean ‘aerosol’ and a subscript 0 to mean spatial average, so $n_{a0}$ in Scales corresponds to $\bar{N}_a$ in the notation used here. Figure 1 of Scales (2004) shows both mean and spatially oscillatory components of aerosol, positive ion, and electron densities for times before heating ($Z_a = 1$, $T_e/T_n = 1$), during heating ($Z_a = 3.5$, $T_e/T_n = 10$), and after heating with aerosol grains still somewhat charged ($Z_a = 1.5$, $T_e/T_n = 1$). The aerosol is assumed immobile (corresponding to the $\varepsilon = 0$ assumption made earlier) so the mean and perturbed aerosol densities are constant and given by $N_a(z) = 0.25n_0(1 + 0.5\cos(k_Bz))$ where $n_0$ is a reference density. Thus, the mean aerosol density is $\bar{N}_a = 0.25n_0$, the amplitude of the perturbed aerosol density is $n_a = 0.125n_0$ and both these quantities remain constant in time. The mean ion density is $\bar{N}_i = n_0$ and is nearly fixed because the ion flux to the dust grains is very small compared to the electron flux (except in the special case when there are almost no electrons, see Sec.4.1.3). Thus, quasineutrality invoked at the time before heating turns on gives $\bar{N}_{e0} = \bar{N}_i - Z_{a0}\bar{N}_a = 0.75n_0$ using $\bar{N}_i = n_0$, $Z_{a0} = 1$, and $\bar{N}_a = 0.25n_0$ so $\Lambda_0 = Z_{a0}\bar{N}_a/\bar{N}_{e0} = 1/3$.

4.1.1 Meta-equilibrium before heating:

Inserting $\Lambda = \Lambda_0 = 1/3$, $n_a = 0.125n_0$, $Z_a = 1$, and $T_e/T_n = 1$ in Eqs.68 and 69 gives $n_e = -3n_0/56 = -0.0536n_0$ and $n_+ = n_0/14 = 0.0714n_0$. Thus, the complete electron density is $N_e(z) = 0.75n_0 - 0.0536n_0\cos(k_Bz)$, the complete positive ion density is $\bar{N}_+(z) = n_0 + 0.0714n_0\cos(k_Bz)$, and the complete aerosol density is $\bar{N}_a(z) = 0.25n_0 + 0.125n_0\cos(k_Bz)$. These complete electron, positive ion, and aerosol densities are in good agreement with the left-hand panels in Fig.1 of Scales (2004).

4.1.2 Meta-equilibrium during heating:

Here $T_e/T_n = 10$ and $Z_a = 3.5$ so using $\Lambda_0 = 1/3$ and $Z_{a0}/Z_a = 1/3.5$ in Eq.72 gives $\Lambda = 7$. Using Eq.1, it is seen that now $\bar{N}_e = Z_a\bar{N}_a/\Lambda = 0.125n_0$ and as expected, $\bar{N}_+ = \bar{N}_e + Z_a\bar{N}_a = n_0$. Using Eq.68 gives $n_e = -0.0054$ which is a factor of ten decrease compared to before heating and using Eq.69 gives $n_+ = 0.432$. Thus, the complete electron density is now $N_e(z) = 0.125n_0 - 0.0054n_0\cos(k_Bz)$, the complete positive ion density is $\bar{N}_+(z) = 0.125n_0 - 0.0054n_0\cos(k_Bz)$, and the complete aerosol density is $\bar{N}_a(z) = 0.25n_0 + 0.125n_0\cos(k_Bz)$.
$n_0 + 0.43n_0 \cos(k_B z)$, and the complete aerosol charge density is $Z_a N_a(z) = 0.875n_0 + 0.4375n_0 \cos(k_B z)$. These complete densities are in good agreement with the middle panels of Fig.1 in Scales (2004). The slight discrepancy is because Scales (2004) allow for a slight reduction in the mean positive ion density as a result of dust charging.

4.1.3 Meta-equilibrium after heating, but with dust retaining some excessive charge:

Here $T_e/T_n = 1$ and $Z_a = 1.5$ so using $\Lambda_0 = 1/3$ and $Z_{a0}/Z_a = 1/1.5$ in Eq.72 gives $\Lambda = 0.6$ at this stage. Using Eq.1, it is seen that now $\bar{N}_e = Z_a \bar{N}_a/\Lambda = 0.625n_0$ and $\bar{N}_+ = \bar{N}_e + Z_a \bar{N}_a = n_0$. Using Eq.68 gives $n_e = -0.072n_0$ and using Eq.69 gives $n_+ = 0.115n_0$. Thus, the complete electron density is now $N_e(z) = 0.625n_0 - 0.072n_0 \cos(k_B z)$, the complete positive ion density is $N_+(z) = n_0 + 0.115n_0 \cos(k_B z)$, and the complete aerosol charge density is $Z_a N_a(z) = 0.375n_0 + 0.188n_0 \cos(k_B z)$. There is now a moderate disagreement with the right hand panels in Fig.1 of Scales (2004); the reason for this is discussed below.

Since the aerosol is supposed to be immobile and is given by $N_a(z) = \bar{N}_a + n_a \cos(k_B z)$ where both $\bar{N}_a$ and $n_a$ are constant in time, Scales’ three lower plots of $Z_a N_a/n_0$ ($Z_{a0}n_0/n_0$ in Scales’ notation) would be expected to have the same ratio of peak-to-peak oscillation amplitude (i.e., $2n_a$) to mean amplitude (i.e., $\bar{N}_a$), if $Z_a$ is the same for all aerosol particles. This is because in Scales’ notation $n_a/n_0$ is being held fixed, the three bottom plots should be of the form $Z_a N_a(z) = Z_a \times (\bar{N}_a + n_a \cos(k_B z))$ so the ratio of the oscillatory part to the mean part should remain constant such that $2n_a/\bar{N}_a = 2 \times 0.125/0.25 = 1$. The ratio of peak-to-peak oscillation amplitude to mean value is unity in the bottom left plot of Scales’ Fig.1, the ratio is somewhat less than one in the bottom middle plot, and it is significantly less than one in the bottom right plot. This discrepancy is likely due to a spatially periodic variation in aerosol charge, i.e., $Z_a = \bar{Z}_a + z_a \cos(k_B z)$ so that $Z_a N_a(z) = (\bar{Z}_a + z_a \cos(k_B z)) (\bar{N}_a + n_a \cos(k_B z)) \simeq \bar{N}_a \bar{Z}_a + (\bar{Z}_a n_a + z_a \bar{N}_a) \cos(k_B z)$ where $z_a$ is negative so that the amount of aerosol charging is anti-correlated with the aerosol density. This is a higher order effect which likely comes about because the change in aerosol charge as calculated in the numerical code in going from the second to third panel is largely due to ion flux on the aerosol (recall that there are almost no free electrons during the heating stage). The ion spatial modulation in the heating stage is both large and correlated with the aerosol density. Since the flux of ions to the aerosol reduces $Z_a$ there would be greater reduction of $Z_a$ at spatial locations where the aerosol density is at a maximum than at locations where the aerosol density is at a minimum. Preferentially attenuating the aerosol charge peaks reduces the peak-to-peak spatial amplitude of the aerosol charge and so corresponds to $z_a$ being nega-
tive. This higher-order effect can be considered to be an imprint on the spatial
distribution of the aerosol charge in the post-heating stage from the imprint
on the ion density profile in the heating stage that in turn was due to the
spatial distribution of the aerosol density before heating.

4.2 Overshoots and undershoots

In the limit that Λ and Λ₀ are both small compared to unity Eqs.68 and 71
give

\[ n_e \simeq -\frac{\Lambda}{\Lambda_0} \frac{n_a Z_{a0}}{\left(1 + \frac{T_e}{T_n}\right)} \] (73)

whereas in the opposite limit where Λ and Λ₀ are both large compared to
unity

\[ n_e \simeq -\frac{n_a Z_{a0}}{1 + \Lambda \frac{T_e}{T_n}} \] (74)

Equations 73 and 74 show that \( n_e \) is an increasing function of \( \Lambda \) if \( \Lambda \ll 1 \), but
\( n_e \) is a decreasing function of \( \Lambda \) if \( \Lambda \gg 1 \).

Consider the standard assumption for heating scenario where (i) \( T_e/T_n \) starts
at unity, (ii) \( T_e/T_n \) is increased to be much greater than unity, and (iii) \( T_e/T_n \)
is then brought back to unity. Combining Eqs.68 and 71 shows that the ratio
of \( |n_e|^2 \) at stage (iii) to what it was in stage (i), i.e., the quantity responsible
for PMSE is,

\[ \frac{|n_e^{(iii)}|^2}{|n_e^{(i)}|^2} = \left(\frac{3 + \Lambda_0 + \frac{2}{\Lambda}}{3 + \Lambda + \frac{2}{\Lambda}}\right)^2 \] (75)

Equation 75 shows that if Λ and Λ₀ are both small compared to unity, then
\( |n_e^{(iii)}/n_e^{(i)}|^2 \sim (\Lambda/\Lambda_0)^2 \) so the stage (iii) PMSE signal would be larger than
the stage (i) signal. On the other hand if Λ and Λ₀ are both large compared
to unity, then \( |n_e^{(iii)}/n_e^{(i)}|^2 \sim (\Lambda_0/\Lambda)^2 \) so the stage (iii) PMSE signal would
be smaller than the stage (i) signal. The former case corresponds to PMSE
overshoot and the latter case to PMSE undershoot. On taking the derivative
of Eq.75 with respect to Λ it is seen that the transition from PMSE signal
increasing with Λ to PMSE signal decreasing with Λ occurs when Λ = \( \sqrt{2} \).

Since Λ is presumed to be larger than Λ₀ after the electrons have been heated
and then returned to their initial temperature, it is seen that undershoot will
occur if Λ₀ > \( \sqrt{2} \).

Hence, overshoot is possible in the limit Λ₀ \ll 1 but HF heating should result
in undershoot for Λ₀ > \( \sqrt{2} \). We note that the possibility of undershoot has
been previously discussed by Biebricher et al. (2006). Overshoot is predicted
to occur only in the region where PMSE is relatively small for the given total
number of electrons. The \( \Lambda = 2 \) situation assumed in Rapp and Lübken (2000) thus lies in a region where undershoot, not overshoot, should occur.

5 Relationship of ambipolar meta-equilibrium state to the Boltzmann model of Havnes et al. (2004)

As noted in the introduction, Havnes et al. (2004) gave a model that did not have any obvious connection to Hill’s ambipolar diffusion equations. We now show that Havnes et al. (2004) is equivalent to the meta-equilibrium state provided \( \Lambda \ll 1 \). If the ambipolar electric field in Eq. 47 is written in terms of an electrostatic potential \( V \), then Eq. 47 becomes

\[
-\nabla V = \frac{\kappa T_e}{q_e} \nabla N_e
\] (76)

and so the electrons have a Boltzmann distribution

\[
N_e(x) = N_e(x_{ref}) \exp \left( -\frac{q_e (V(x) - V(x_{ref}))}{\kappa T_e} \right)
\] (77)

where \( x_{ref} \) is some reference position. In steady state, the ion flux vanishes and so Eq. 63 together with Eq. 76 give

\[
\nabla N_+ = -\frac{q_+ N_+}{\kappa T_n} \nabla V
\] (78)

in which case the ions also have a Boltzmann distribution

\[
N_+(x) = N_+(x_{ref}) \exp \left( -\frac{q_+ (V(x) - V(x_{ref}))}{\kappa T_n} \right).
\] (79)

The question is where \( x_{ref} \) is chosen. Havnes et al. (2004) choose \( x_{ref} \) to be outside the aerosol region so that \( N_i(x_{ref}) = N_e(x_{ref}) \). However, inside the aerosol region, \( N_e \) will be much smaller than \( N_i \) if there is significant charge on the dust. In such a case \( |e (V(x) - V(x_{ref}))|/\kappa T_e| \) cannot be small so it is not permissible to make a small argument Taylor expansion of the exponential functions as Havnes et al. (2004) did when they went from their Eq.(3) to their Eq.(13). If \( x_{ref} \) is chosen to be inside the aerosol cloud at a location where \( N_+(x_{ref}) = \bar{N}_+, N_e(x_{ref}) = \bar{N}_e \) and the potential zero-point is chosen so that \( V(x_{ref}) = 0 \), then it is seen using Eq.65 that
\[ N_e(x) = \bar{N}_e \exp \left( -\frac{q_e V(x)}{\kappa T_e} \right) \]  \hspace{1cm} (80)

\[ N_+(x) = (1 + \Lambda) \bar{N}_e \exp \left( -\frac{q_+ V(x)}{\kappa T_n} \right). \]  \hspace{1cm} (81)

The quasineutrality condition becomes

\[ \bar{N}_e \exp \left( -\frac{q_e V(x)}{\kappa T_e} \right) + Z_a N_a = (1 + \Lambda) \bar{N}_e \exp \left( -\frac{q_+ V(x)}{\kappa T_n} \right), \]  \hspace{1cm} (82)

i.e.,

\[ (1 + \Lambda) \exp \left( -\frac{q_+ V(x)}{\kappa T_n} \right) - \exp \left( -\frac{q_e V(x)}{\kappa T_e} \right) = \Lambda \]  \hspace{1cm} (83)

which differs significantly from Eq. (3) of Havnes et al. (2004) and so will not lead to their Eq. (13) and following equations (note that \( s \) in Havnes et al. (2004) corresponds to \( \Lambda \)). Thus, the quasineutrality condition used in Havnes et al. (2004) is only valid if \( \Lambda \ll 1 \) and so is not consistent with the \( \Lambda = 2 \) assumption used in Rapp and Lübken (2000).

Equations 80, 81 and the quasineutrality relation Eq. 83 are equivalent to the meta-equilibrium discussed here. This equivalence is demonstrated as follows:

Using \( q_e = - q_+ \) and eliminating \( q_e V \) between Eqs. 80 and 81 gives

\[ \left( \frac{N_e(x)}{N_e} \right)_{T_e/T_n} \frac{N_+(x)}{(1 + \Lambda) N_e} = 1 \]  \hspace{1cm} (84)

or on using Eqs. 2, 3, and 65

\[ \left( \frac{\bar{N}_e + n_e}{N_e} \right)_{T_e/T_n} \frac{(1 + \Lambda) \bar{N}_e + n_+}{(1 + \Lambda) N_e} = 1 \]  \hspace{1cm} (85)

which reduces to Eq. 66 upon linearizing.

6 Discussion and Conclusions

Because the ratio of aerosol diffusivity to positive ion diffusivity is of order \( \varepsilon \) where \( \varepsilon = 10^{-5} - 10^{-3} \), consideration of the coupled diffusion equations shows that the system relaxes on the fast time scale to a long-lived meta-equilibrium. The electron density relevant to PMSE in this meta-equilibrium is independent of the initial condition for the electron density but rather depends on the aerosol initial condition and on state parameters such as electron temperature, aerosol charge, and the ratio of the number of electrons deposited on
aerosol particles to the number of free electrons. Identification of this property enables determination of the PMSE-relevant electron density inhomogeneity for time-dependent electron temperature and time-dependent aerosol charge using simple algebraic relations because the solution of the rather complex time-dependent equations is a fast relaxation to the meta-equilibrium.

Because Bragg backscattering, the physical mechanism responsible for PMSE, is a coherent mechanism due to the constructive interference of a large number of in-phase scattering elements, there must be a large number of scattering elements, a property inherently missing from previously used single Gaussian bump models. The criticism by La Hoz et al. (2006) that Rapp and Lübken (2000) used the wrong width for a single Gaussian bump thus becomes moot because Gaussian bumps do not appropriately characterize the electron spatial inhomogeneity responsible for Bragg backscattering.

The meta-equilibrium Bragg component electron density is shown to be always anti-correlated with the aerosol density no matter what initial condition is assumed for the electron density. Because this electron density is an imprint of the aerosol Bragg component, for times longer than the aerosol Bragg component diffusion time, all quantities go to zero and so PMSE could not occur. One can think of the electron density spatial profile as being proportional to the photographic negative of the aerosol spatial profile. This raises the issue of whether neutral air turbulence is responsible for the aerosol density inhomogeneity leading to PMSE because PMSE has been observed in situations where there has been no neutral air turbulence for times longer than the 10-30 s aerosol characteristic diffusion time.

The equations leading to the meta-equilibrium state can be interpreted intuitively as being analogous to equations describing a fictitious particle in a two-dimensional space whereby the particle falls down a hill to come to rest in a valley while being constrained to a straight-line track. Different choices of electron initial conditions correspond to assuming the particle starts at different points on the track, but since the resting point is always at the bottom of the valley it does not matter where the particle starts on the track.

A comparison between the predictions of the analytic model presented here and particle-in-cell numerical calculations by Scales (2004) shows reasonable agreement. The slight disagreement in the post-heating recovery situation is explained as due to the development of a periodic spatial modulation of $Z_a$ resulting from the spatial modulation of positive ions in the heating phase. This is because the recovery of $Z_a$ to its pre-heating value is due to ion flux to the aerosol grains so that the recovered $Z_a$ will transiently incorporate any spatial modulation of this ion flux. The ion density during the heating phase has a strong spatial modulation and so is responsible for the ion flux having a spatial modulation.
The identification of the meta-equilibrium state enables straightforward generalization to the situation where electron heating causes the aerosol charge to increase and the number of free electrons to decrease. Consideration of this situation shows that after electron heating has been turned on and then off, the PMSE strength will be larger than its initial value if the Havnes parameter \( \Lambda_0, \Lambda \ll 1 \) (i.e., overshoot), but the PMSE strength will be smaller than its initial value if \( \Lambda_0, \Lambda \gg 1 \) (i.e., undershoot).

The Boltzmann approach used by Havnes et al. (2004) has been shown to be equivalent to the meta-equilibrium state derived here if \( \Lambda \ll 1 \), but if \( \Lambda \) is of order unity or larger, this Boltzmann approach fails to satisfy quasi-neutrality.

Finally, it is worthwhile noting that numerical integration methods are fundamentally incapable of proving existence of a meta-equilibrium state. This is because numerical methods only calculate one specific case at a time and so cannot prove a general principle. Demonstrating via numerical integration that \( N \) different choices of electron density initial condition all lead to the same meta-equilibrium, does not prove that a different \( N+1^{th} \) choice for electron density initial condition will also lead to the same meta-equilibrium or even to any meta-equilibrium. Universality can only be proved by deriving an analytic expression that demonstrates existence of the meta-equilibrium state, i.e., by deriving Eq.68.

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Appendix A- Detailed demonstration that the time-dependent normal mode solutions relax to meta-equilibria when the electron temperature is time-dependent

For times \( t_f \ll t \ll t_s \), the solution to Hill’s equations, Eq.26, reduces to

\[
\psi \simeq \psi_0 - \frac{F}{\gamma_f} \tag{86}
\]

or

\[
\Delta \psi = \psi - \psi_0 = - \frac{F}{\gamma_f}. \tag{87}
\]

Let us consider the implications of Eq.87 for Rapp and Lübken (2000)’s initial conditions \( n_e = n_0 \) and \( n_\oplus = (1 + 2\Lambda) n_0 \) with \( T_e^c \) changing from the cold temperature \( T_e^c = T_n \) to the hot temperature \( T_e^h \) and then back again. We assume \( \Lambda = 2 \) to be consistent with Rapp and Lübken (2000).

First stage: Here the initial conditions are \( n_e = n_0, n_\oplus = (1 + 2\Lambda) n_0, \) and
\[ T_e^c/T_n = 1 \] in which case
\[ F = 4(\Lambda + 1)n_0 \]
\[ \gamma_f = 2(\Lambda + 2). \]  

Thus, Eq.87 gives
\[ \Delta \psi = \frac{F}{\gamma_f} = -\frac{2(\Lambda + 1)n_0}{(\Lambda + 2)} \]  
in which case for \( t \gg t_f \), Eq.86 gives
\[ n_e = n_e(0) + \Delta \psi = -\frac{\Lambda}{\Lambda + 2}n_0 \]  
\[ n_\oplus = n_\oplus(0) + \Delta \psi = \frac{2\Lambda + 3}{\Lambda + 2}n_0. \]  

In particular \( n_e = -\frac{1}{2}n_0 \) and \( n_\oplus = 7n_0/2 \) if \( \Lambda = 2 \). This shows that the specified initial conditions do not constitute a relaxed state even though they satisfy quasineutrality and furthermore that the system will relax in milliseconds to the Eq.91, 92 state which will persist for long times (until \( t \) becomes of order \( t_s \)). In particular, this shows that when the initial condition is such that the electrons are correlated with the aerosols (positive \( n_e \) initial condition), they quickly relax to the preferred anti-correlated state (negative \( n_e \) as given by Eq.91). Furthermore, the ratio of the relaxed electron density state to the initial state scales as \(-\Lambda/(\Lambda + 2)\); this leads to Cho’s misconception that PMSE requires \( \Lambda \) to be larger than unity.

Second stage: The initial conditions for the second stage are given by Eqs.91 and 92 and now \( T_e^c = T_e^h \) where \( T_e^h/T_n \gg 1 \). We reset the clock so \( \tau = 0 \). Because the electrons are now hot, Eqs.19 and 25 give
\[ F = 2\Lambda (1 + \Lambda)\left(1 - \frac{T_e^h}{T_n}\right)n_0 \]  
\[ \gamma_f = 2\left(1 + \frac{T_e^h}{T_n}(1 + \Lambda)\right) \]  
so now Eq.87 gives
\[ \Delta \psi \approx \frac{\Lambda}{\Lambda + 2}\left(1 - \frac{T_n}{T_e^h}\left(\frac{\Lambda + 2}{\Lambda + 1}\right)\right)n_0. \]  

Using the initial conditions given by Eqs.91 and 92, Eq.86 then gives
\[ n_e \approx n_0(0) + \Delta \psi \approx -\frac{1}{T_e^c/T_n}\frac{\Lambda}{\Lambda + 1}n_0 \]  
\[ n_\oplus = n_\oplus(0) + \Delta \psi \approx \left(2 - \frac{1}{T_e^c/T_n}\frac{1}{\Lambda + 1}\right)\Lambda n_0. \]
Third stage: The electron temperature reverts to $T_e/T_n = 1$. We reset the clock and the initial conditions are now given by Eqs. 96 and 97. Also $\gamma_f$ reverts to the value prescribed by Eq. 89 and $F \approx 2\Lambda n_0$. 

Thus,

$$\Delta \psi = -\frac{F}{\gamma_f} = -\frac{\Lambda}{(\Lambda + 2)} n_0$$

(99)

and so Eq. 86 gives

$$n_e = n_0(0) + \Delta \psi = -\frac{\Lambda}{\Lambda + 2} n_0$$

(100)

$$n_{\oplus} = n_\oplus(0) + \Delta \psi = \Lambda \left(\frac{2\Lambda + 3}{\Lambda + 2}\right) n_0$$

(101)

which is the same as the end of the first stage. Equations 91, 96, and 100 were determined by taking into account the time-dependent temperature in detail (i.e., invoking initial conditions and keeping track of the fast decay). However, these three equations each arrive at the meta-equilibrium given by Eq. 28 using $n_a = \Lambda n_0$. Thus, it is not necessary to follow the detailed time-dependence because the system always relaxes to the meta-equilibrium given by Eq. 28 and this just depends on $n_a$, $\Lambda$, and the current $T_e/T_n$ ratio.

References


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